JOURNAL OF APPROXIMATION THEORY 17, 135–149 (1976)

Constrained Approximation and Hermite Interpolation with Smooth Quadratic Splines: Some Negative Results

JAMES W. DANIEL*

Departments of Mathematics and of Computer Sciences and Center for Numerical Analysis, University of Texas at Austin, Austin, Texas 78712

Communicated by Richard S. Varga

Received October 9, 1974

Quadratic splines are generated which interpolate a function and its derivative at points midway between alternate pairs of knots, and the error bound is shown to be precisely of order h^2 rather than h^3 as expected. This result is related to best constrained approximation by splines and thence to Galerkin methods for constrained problems.

I. INTRODUCTION

Galerkin or Rayleigh–Ritz methods have become very popular again for the solution of various practical problems posed as an equation, variational equality, or the lozation of a point x^* minimizing some functional f over some *linear* space X (Schultz [16], Strang–Fix [19]). In this last setting, for example, one approximates x^* by a point x_n^* minimizing f over some linear space X_n of approximations. In many specific cases and in fair generality, it can be shown that the error between x_n^* and x^* is of the same order as the error in *best approximation* of x^* by elements of X_n ; one then applies results from approximation theory to give a priori error bounds on $x^* - x_n^*$.

More recently, attempts have been made, with some success, to extend the above-mentioned methods and results to more general problems posed as a variational *in*equality or as the location of a point x^* minimizing a functional f over a *convex subset* C of a linear space X (Aubin [1], Bosarge *et al.* [5, 6], Daniel [9], Falk [11], Mosco-Strang [14], Strang [17, 18]). Again, in this latter setting, one approximates x^* by x_n^* minimizing f over some convex subset C_n of a linear space X_n of approximations. In a very limited number of cases it

* Research supported in part by the United States Office of Naval Research under Contract N00014-67-A-0126-0015, NR0-044-425; reproduction in whole or in part is permitted for any purposes of the United States government.

has been shown that the error $x^* - x_n^*$ is of the order of the sum of the best approximation errors of x^* by elements of C_n (not X_n) and of x_n^* by elements of C. For use now in these cases and, hopefully, later in more generality, one therefore desires approximation-theory results on the order of the error in these constrained best approximations for various useful approximating spaces X_n.

It is this interest in the Galerkin method for constrained problems that leads us in this paper to consider a problem involving constrained best approximation by spline functions. Some results in this direction have been presented or discussed by Strang (Strang [17, 18], Mosco–Strang [14]). In line with his approach, we consider the approximation on [0, 1] of a nonnegative function f from below by nonnegative splines s; thus s is constrained to satisfy $0 \leqslant s(t) \leqslant f(t)$ for $0 \leqslant t \leqslant 1$ and we wish to bound the least error $f - s$. Under the partial order $\prec$ defined by $s_1 \prec s_2$ if and only if $s_1(t) \leqslant s_2(t)$ for all t in [0, 1], it is natural to consider a *maximal* spline s^* subject to $0 \prec s^* \prec f$. In $L_2(0, 1)$ it is clear that a constrained best approximation must be maximal, while in $C[0, 1]$ it is clear that, given a constrained best approximation $\bar{s}$, one can find a maximal constrained best approximation s^* satisfying $\bar{s} \prec s^*$; this is why we say the consideration of maximal splines is "natural." Strang has shown (Strang [17, 18], Mosco–Strang [14]) that a maximal first degree spline (piecewise *linear* polynomial) approximates reasonable functions f to the optimal order of best *unconstrained* approximation; one simply deduces that for s^* to be maximal it must interpolate the values of f (and also of its derivative) at a certain set of points, enough to allow one to use results on piecewise linear interpolation errors in order to derive the desired bounds.

When we consider the use of splines of degree greater than one, the picture changes somewhat. Certainly if s^* is maximal and if B is any nontrivial nonnegative spline then we cannot have $s^* + \lambda B \prec f$ for any postive λ; this usually allows us to conclude that s^* interpolates f at some ponits in the support of B. By choosing for B the extreme points of the cone of nonnegative splines (Burchard [7]), we can indeed conclude that a maximal spline s^* must interpolate f (and its derivative) at a large number of points distributed rather uniformly over [0, 1]. Following the line of the argument for splines of degree one, we would next expect to quote some known results on the accuracy of spline interpolation to give bounds on our best constrained-approximation error; unfortunately such error estimates do not appear to be known (except in very special cases not applicable here (Varga [21])). The difficulty is that the interpolation points, while distributed evenly throughout [0, 1], need not fall at the spline's knots or at other special points for which the resulting error bounds are known; in addition, our case involves Hermite interpolation (that is, for both f and f') with very smooth splines (say continuously differentiable quadratics) rather than with the rougher splines usually associated with Hermite interpolation. Thus we are led to consider the question of the accuracy of Hermite interpolation by very smooth splines at various points in [0, 1].

To study the Galerkin method for constrained problems one naturally then can consider a sequence of related questions: (1) What is the order of the error in constrained best approximation by splines? (2) What is the order of the error in approximation by maximal splines? (3) What is the order of the error in approximation by smooth splines interpolating f and f' at various points? The results of this paper are essentially "negative" results relating to questions (2) and (3). We show in Section 2 that an Hermite interpolation using reasonable interpolation points and smooth splines of degree two (piecewise quadratic polynomials) leads to second-order accuracy rather than third-order as usually (de Boor [2]) associated with second-degree splines; such behavior is somewhat unexpected (Cox [8]). In Section 3 we exhibit a maximal second-degree spline approximation which also has an error of second order rather than the expected third order. We have *not*, however, been able to show that the best constrained-approximation error by second-degree splines is of second order; our results merely indicate that one cannot extend Strang's maximal-spline arguments in order to prove third-order accuracy. Some experimental computational results seem to indicate that the errors are indeed of third order.

For the convenience of the reader, the extremely tedious details of our computations of the errors described above have been placed in the Appendix rather than in the main body of the paper. Another approach to these examples will appear in de Boor [4].

2. Interpolation Schemes for Second-Degree Splines

We consider second-degree splines s in $C^1[0, 1]$ with uniformly spaced knots at $t_i = ih$ for $0 \leqslant i \leqslant N$, where $h = 1/N$. Although there are a variety of ways to represent (and solve for) s, we choose to represent s as $s = \sum_{i=-1}^{N} a_i B_i$ in terms of the B-splines $\{B_i\}_{-1}^{N}$. The B-spline B_i is nonnegative and is given by

$$\begin{aligned}
B_i(x) &= 0 \qquad \text{for} \quad x \leqslant (i-1)h \\
&= (1/2h^2)[x-(i-1)h]^2 \qquad \text{for} \quad (i-1)h \leqslant x \leqslant ih \\
&= (1/2h^2)[x-(i-1)h]^2 - (3/2h^2)[x-ih]^2 \quad \text{for } ih \leqslant x \leqslant (i+1)h \\
&= (1/2h^2)[x-(i+2)h]^2 \qquad \text{for} \quad (i+1)h \leqslant x \leqslant (i+2)h \\
&= 0 \qquad \text{for} \quad (i+2)h \leqslant x.
\end{aligned}$$

In particular, we have the following values for B_i and B_i':

x	$(i-1)h$	$(i-\frac{1}{2})h$	ih	$(i+\frac{1}{2})h$	$(i+1)h$	$(i+\frac{3}{2})h$	$(i+2)h$
$B_i(x)$	0	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{8}$	0
$B_i'(x)$	0	$\frac{1}{2h}$	$\frac{1}{h}$	0	$-\frac{1}{h}$	$-\frac{1}{2h}$	0

Since B_i is nonnegative and $\sum_{i=-1}^{N} B_i(x) \equiv 1$, it follows easily that $\| s \|_\infty = \| \sum_{i=-1}^{N} a_i B_i \|_\infty \leqslant \max_i | a_i | \, \| \sum_{i=-1}^{N} B_i \| = \|(a_{-1}, a_0, ..., a_N)\|_\infty$; it is also trivial to see that $\| s' \|_\infty \leqslant (2/h) \|(a_{-1}, a_0, ..., a_N)\|_\infty$. When analyzing an interpolation scheme, as usual (de Boor [3]) we notice that it defines a linear projection Π_N from $C^r[0, 1]$ into a spline subspace S_N of $C^p[0, 1]$, and then we use Lebesgue's inequality to estimate $\| \Pi_N f - f \| = \| \Pi_N f - \Pi_N s + s - f \| = \|(\Pi_N - I)(f - s)\| \leqslant \| \Pi_N - I \| \, \| f - s \|$ for all s in S_N; thus $\| \Pi_N f - f \|$ is no larger than $\| \Pi_N - I \| \leqslant 1 + \| \Pi_N \|$ times the (usually known) error of best approximation by elements of S_N, and we need only estimate $\| \Pi_N \|$. In practize the interpolation is usually defined in terms of the B-spline coefficients $a_{-1}, ..., a_N$ by a system of equations $A_N a = p_N f$, where $a^T = (a_{-1}, ..., a_N)$ and p_N maps f into the vector of interpolated values and is uniformly bounded, for example, with $\| p_N \| \leqslant 1$. Then we have $\| \Pi_N f \|_\infty = \| \sum_{i=-1}^{N} a_i B_i \|_\infty \leqslant \| a \|_\infty = \| A_{-1}^N p_N f \|_\infty \leqslant \| A_{-1}^N \|_\infty \| f \|$ and we need only bound $\| A_{-1}^N \|_\infty$. Such estimates will form the heart of our subsequent analyses.

Now we move on to consider interpolation in our $(N + 2)$-dimensional space of splines of degree two. It is well known (de Boor [2]) that for f in $C^3[0, 1]$ the error of *best approximation* is of the order of h^3 and, more precisely, that $\| f - s \|_\infty + h \| f' - s' \|_\infty = \mathcal{O}(h^3)$ for some spline s; our concern here, however, is with the errors resulting from *interpolation schemes* with various patterns of interpolation points, for which the problems of existence and uniqueness of the interpolants are well understood (Schoenberg–Whitney [15]), but for which the problems of sharp error bounds have been less thoroughly treated. In fact, the relationships among the pattern of interpolation points, the placement of knots, the smoothness of the interpolated function, and the resulting errors are quite subtle. The quadratic spline Q interpolating the values of f at $t_i = ih$ for $0 \leqslant i \leqslant N$ and of f' at $t_0 = 0$, for example, satisfies $\| f - Q \|_\infty \leqslant \text{const} \times h^{-1} \times \| f - s \|_\infty$ for all quadratic splines s, so that $\| f - Q \|_\infty = \mathcal{O}(h^2)$ for f in $C^3[0, 1]$ rather than $\| f - Q \|_\infty = \mathcal{O}(h^3)$ as expected. If f is still smoother, however, arguments more subtle than use of Lebesgue's inequality can be used to show that $\| f - Q \|_\infty$ is in fact of third order in h. Our justification of this last assertion, apparently first made by Daniel [10], was by tedious direct computations similar to those in this present paper; since this result is not central to the present work, however, and since a much more concise and thorough treatment will appear in de Boor [4], where the bound $\|f - Q\|_\infty \leqslant \text{const} \times h^3 \times [\|f^{(3)}\|_\infty + \operatorname{Var}(f^{(3)})]$ is derived, we pursue this no further now.

In contrast to the complex situation above for interpolation at the knots, the simple scheme (Subbotin [20]) of interpolating the values of f at the points $t = 0$, $t = (i + \frac{1}{2})h$ for $0 \leqslant i \leqslant N - 1$, and $t = 1$, however, yields $\|f - Q\|_\infty \leqslant \text{const} \times \|f - s\|_\infty$ for all splines s so that the error is of order h^3 for f in $C^3[0, 1]$, for example (Marsden [13], Kammerer–Reddien–Varga [12]). Recall, however, that our motivation in Section 1 for studying interpolation schemes concerned maximal splines $s \prec f$, so that s' and f' must agree whenever s and f do. We naturally wonder therefore what errors result from such Hermite interpolation at various points such as the knots or the midpoints in comparison with what we have described above for simple Lagrange interpolation with these patterns of points; of course, to avoid demanding too much of our $(N + 2)$-dimensional space, we can perform our Hermite (double) interpolation at only about half as many points as for simple Lagrange interpolation.

First, to show that we *can* get good error bounds via Hermite interpolation with our smooth splines rather than the rougher splines usually associated with Hermite interpolation, we show that an error of the optimal order h^3 results from the simple scheme of Hermite interpolation at every other *knot*. To see this, suppose that $N = 2M$ and that we require the $2(M + 1) = N + 2$ conditions $s(2ih) - f(2ih) = s'(2ih) - f'(2ih)$ for $i = 0, 1, \ldots, M$. This, in fact, defines a purely local scheme in that the values of $s(x)$ for $2ih \leqslant x \leqslant (2i + 2)h$ are determined only by f and f' at $2ih$ and $(2i + 2)h$. The conditions at $2ih$ require $(1/2)\, a_{2i-1} + (1/2)\, a_{2i} = f(2ih)$ and $-(1/h)\, a_{2i-1} + (1/h)\, a_{2i} = f'(2ih)$ so that $a_{2i-1} = f(2ih) - (1/2)\, hf'(2ih)$, $a_{2i} = f(2ih) + (1/2)\, hf'(2ih)$. From this we can easily find the order of the error. We know that for f in $C^3[0, 1]$ there is some quadratic spline s_0 such that $\|s_0 - f\|_\infty = \mathcal{O}(h^3)$ and $\|s_0' - f'\|_\infty = \mathcal{O}(h^2)$, so that $\|s_0 - f\| = \mathcal{O}(h^3)$ where $\|g\| = \|g\|_\infty + h\|g'\|_\infty$. Therefore, as we saw at the start of this section, our interpolation scheme gives an error $\|s - f\| \leqslant (1 + \|\Pi_N\|) \cdot \mathcal{O}(h^3)$ where Π_N is the projection defined by our scheme. From the above formulas for a_{2i} and a_{2i-1} we see that $\|(a_{-1}, a_0, \ldots, a_N)\|_\infty \leqslant \|f\|_\infty + (1/2)h\|f'\|_\infty \leqslant \|f\|$; since $\|s\|_\infty \leqslant \|(a_{-1}, a_0, \ldots, a_N)\|_\infty$ and $\|s'\|_\infty \leqslant (2/h)\|(a_{-1}, a_0, \ldots, a_N)\|_\infty$, we have $\|s\| \leqslant 3\|(a_{-1}, a_0, \ldots, a_N)\|_\infty$ and hence $\|s\| = \|\Pi_N f\| \leqslant 3\|f\|$. Therefore, for f in $C^3[0, 1]$, our Hermite interpolation scheme at alternate knots comes within a multiple of three of obtaining best approximation error, and in particular approximates f to $\mathcal{O}(h^3)$ and f' to $\mathcal{O}(h^2)$ accuracy. This accuracy should be compared with the accuracy of order h^2 obtained from simple interpolation at the knots for f

merely known to lie in $C^3[0, 1]$. To obtain the higher optimal order accuracy h^3 from simple interpolation we needed either to have a smoother function f or to shift the interpolation points to midway between the knots; we will next show that for Hermite interpolation, however, this shift to the midpoints yields only accuracy of order h^2, and that this does *not* improve for smoother functions. It would be interesting to know precisely when the optimal errors occur in terms of the smoothness of f, the pattern of the knots, and the pattern and multiplicities of the interpolation points; some results in this direction, including discussions of our present examples from the different viewpoint of de Boor [3], will be found in [4].

We now consider Hermite interpolation at midpoints rather than knots; the question arises as to precisely how to do this. If, for example, $N = 2M$ is even, then we might interpolate twice at each of the M points $(2i + \frac{1}{2})h$ for $0 \leqslant i \leqslant M - 1$; the two remaining degrees of freedom could be specified by interpolation also at the endpoints zero and one. On the other hand, if $N = 2M + 1$ is odd, then we might interpolate twice at each of the $M + 1$ points $(2i + \frac{1}{2})h$ for $0 \leqslant i \leqslant M$; the one remaining degree of freedom might be specified by interpolation at one endpoint.

To allow us to consider both the above cases and a later generalization at once, we suppose that $N = 2M + 1$ and that we interpolate twice at each of the M points $(2i + \frac{1}{2})h$ for $1 \leqslant i \leqslant M$, leaving three degrees of freedom. If we further interpolate twice at $\frac{1}{2}h$ and once at 0 or 1 we get the second case above; however, if instead we further interpolate twice at $\frac{1}{2}h$, once at each 0 and $1 - h$, and not at $(2i + \frac{1}{2})h$ for $i = M$, and only consider the interval $[0, 1 - h]$ we get the first case above.

We now try to find the B-spline coefficients for the above interpolation scheme. From $s((2i + \frac{1}{2})h) = g_i \equiv f((2i + \frac{1}{2})h)$ for a given function f for $1 \leqslant i \leqslant M$, we obtain $a_{2i-1}(\frac{1}{8}) + a_{2i}(\frac{3}{4}) + a_{2i+1}(\frac{1}{8}) = g_i$, while from $s'((2i + \frac{1}{2})h) = g_i' \equiv f'((2i + \frac{1}{2})h)$ for $1 \leqslant i \leqslant M$, we obtain $a_{2i-1}(-1/2h) + a_{2i}(0) + a_{2i+1}(1/2h) = g_i'$. We think of a_{-1}, a_0, and a_1 as representing our three degrees of freedom and then try to solve for a_j for $2 \leqslant j \leqslant N$ in terms of a_{-1}, a_0, and a_1. From $a_{2i+1} = a_{2i-1} + 2hg_i'$ for $1 \leqslant i \leqslant M$, we trivially obtain

$$a_{2i+1} = a_1 + 2h \sum_{j=1}^{i} g_j' \qquad \text{for} \quad 1 \leqslant i \leqslant M. \tag{2.1}$$

Substituting Eq. (2.1) into the relation $a_{2i-1} + 6a_{2i} + a_{2i+1} = 8g_i$ obtained above for $1 \leqslant i \leqslant M$ yields

$$a_{2i} = -\frac{1}{3} a_1 + \frac{4}{3} g_i - \frac{2h}{3} \sum_{j=1}^{i-1} g_j' - \frac{h}{3} g_i' \qquad \text{for} \quad 1 \leqslant i \leqslant M. \tag{2.2}$$

If, as in the case above for Hermite interpolation at every other knot, we define $\| f \| = \| f \|_\infty + h \| f' \|_\infty$, it follows easily from Eqs. (2.1) and (2.2) that $\|(a_{-1}, \ldots, a_N)\|_\infty \leqslant \|(a_{-1}, a_0, a_1)\|_\infty + (\frac{4}{3} + h^{-1}) \| f \|$. If we choose a complete interpolation scheme for which $\|(a_{-1}, a_0, a_1)\|_\infty \leqslant (c_1 + c_2 h^{-1}) \| f \|$ for some constants c_1 and c_2, then we will have bounded the norm $\| \Pi_N \|$ for our projection by $\mathcal{O}(h^{-1})$ and would therefore conclude that our spline interpolant has at least second-order accurate function values rather than third as expected. In fact, if we complete our scheme by interpolating twice at $(1/2)h$ and once at 0 we find $a_1 = a_{-1} + 2hf'((1/2)h)$, $a_0 = -(1/3)\, a_{-1} + (4/3) f((1/2)h) - (h/3) f'((1/2)h)$, and $a_{-1} = 3f(0) - 2f((1/2)h) + (h/2) f'((1/2)h)$ so that $\|(a_{-1}, a_0, a_1)\|_\infty \leqslant 7 \| f \|$ as desired; if we complete our scheme by interpolating twice at $(1/2)h$, once each at 0 and $1 - h$, and not at $(2i + (1/2))h$ for $i = M$, and then consider the approximation on $[0, 1 - h]$, we of course find the same formulas for a_{-1}, a_0, a_1, but the formula for a_{2m} changes slightly, yielding $\|(a_{-1}, a_0, a_1)\|_\infty + (2 + h^{-1}) \| f \|$. A tedious calculation in the Appendix shows that for the function $f(t) \equiv t^3$ the $\mathcal{O}(h^2)$ error estimates are sharp in that for each of these interpolation schemes we have $\| f - s \|_\infty \geqslant ch^2$ for some $c > 0$.

We summarize these results.

THEOREM 2.3. *For $N = 2M + 1$ let s be a quadratic spline with knots at ih for $0 \leqslant i \leqslant N$ and satisfying $s((2i + \frac{1}{2})h) - f((2i + \frac{1}{2})h) = s'((2i + \frac{1}{2})h) - f'((2i + \frac{1}{2})h) = 0$ for $0 \leqslant i \leqslant M$ with $(2M + 1)h = 1$. A unique such spline s exists also satisfying $s(0) = f(0)$ [alternatively, $s(1) = f(1)$] and then we have $\| s - f \|_\infty + h \| s' - f' \|_\infty \leqslant ((25/3) + h^{-1})[\| \sigma - f \|_\infty + h \| \sigma' - f' \|_\infty]$ for all quadratic splines σ on this uniform mesh; in particular, if f is in $C^3[0, 1]$, then $\| f - s \|_\infty = \mathcal{O}(h^2)$, $\| f' - s' \|_\infty = \mathcal{O}(h)$. This bound is sharp in that, for the function $f(x) \equiv x^3$ [alternatively, $f(x) = (1 - x)^3$], there is a $c > 0$ such that $\| s - f \|_\infty \geqslant ch^2$ for small h no matter what value is chosen for the free parameter $s(0)$.*

THEOREM 2.4. *For $N = 2M$ let s be a quadratic spline with knots at ih for $0 \leqslant i \leqslant N$ and satisfying $s((2i + \frac{1}{2})h) - f((2i + \frac{1}{2})h) = s'((2i + \frac{1}{2})h) - f'((2i + \frac{1}{2})h) = 0$ for $0 \leqslant i \leqslant M - 1$ with $2Mh = 1$. A unique such spline s exists also satisfying $s(0) - f(0) = s(1) - f(1) = 0$, and we then have $\| s - f \|_\infty + h \| s' - f' \|_\infty \leqslant (9 + 2h^{-1})[\| \sigma - f \|_\infty + h \| \sigma' - f' \|_\infty]$ for all quadratic splines σ on this uniform mesh; in particular, if f is in $C^3[0, 1]$, then $\| f - s \|_\infty = \mathcal{O}(h^2)$ and $\| f' - s' \| = \mathcal{O}(h)$. This bound is sharp in that for the function $f(x) \equiv x^3$, there is a $c > 0$ such that $\| f - s \|_\infty \geqslant ch^2$ for small h.*

3. An $\mathcal{O}(h^2)$-Accurate Maximal Quadratic Spline

As indicated in Section 1, it is of considerable interest to understand the error in approximation by maximal splines. That is, if a nonnegative function f is given on [0, 1] and if s_0 is a spline satisfying $0 \leqslant s_0(x) \leqslant f(x)$ on [0, 1] such that $s_0(x) \leqslant s(x) \leqslant f(x)$ on [0, 1] for a spline s implies $s_0 = s$, then what is the order of the error $f - s_0$? We consider a special case of this problem.

Let $f(x) \equiv x^3$, let $N = 2M + 1$, and consider the above question for maximal quadratic splines s_0 with knots at ih for $0 \leqslant i \leqslant N$, where $h = 1/N$. As mentioned in Section 1, one might hope that $\| s_0 - f\|_\infty = \mathcal{O}(h^3)$, thus showing that constrained best approximation in this case gives the same order of error as unconstrained best approximation; we show in this section that in fact one need not have $\| s_0 - f\|_\infty = \mathcal{O}(h^3)$.

On the interval $[0, h]$, s_0 is a quadratic polynomial and must satisfy $0 \leqslant s_0(x) \leqslant x^3$; clearly then $s_0(x) \equiv 0$ in $[0, h]$ and therefore we must have $a_{-1} = a_0 = a_1 = 0$ in the B-spline representation $s_0(x) = \sum_{i=-1}^{N} a_i B_i(x)$. We now consider a specific spline s_0 which we will show to be maximal: Let s_0 be that unique quadratic spline with $a_{-1} = a_0 = a_1 = 0$ and satisfying $s_0((2i + \frac{1}{2})h) - f((2i + \frac{1}{2})h) = s_0'((2i + \frac{1}{2})h) - f'((2i + \frac{1}{2})h) = 0$ for $1 \leqslant i \leqslant M$. From our work in Section 2 we know that Eqs. (2.1) and (2.2) are valid. Since $a_1 = 0$ and $f(x) = x^3$, we obtain

$$a_{2i+1} = 6h^3 \sum_{j=1}^{i} (2j + \tfrac{1}{2})^2 \qquad \text{for} \quad 1 \leqslant i \leqslant M, \tag{3.1}$$

$$a_{2i} = \tfrac{4}{3}h^3(2i + \tfrac{1}{2})^3 - h^3(2i + \tfrac{1}{2})^2 - 2h^3 \sum_{j=1}^{i-1} (2j + \tfrac{1}{2})^2 \qquad \text{for} \quad 1 \leqslant i \leqslant M. \tag{3.2}$$

In the Appendix we show that this leads immediately to the following result.

Lemma 3.3. *Let s_0 be the quadratic spline constructed above interpolating f. Then $f(1) - s_0(1) = (1/12)\, h^2 + (17/24)\, h^3$ and $\|f - s_0\|_\infty$ is precisely of order h^2 as h tends to zero.*

By using the explicit representations of s_0 in each interval, we also show in the Appendix that the following holds.

Lemma 3.4. *Let s_0 be the quadratic spline constructed above interpolating f. Then for all x in [0, 1] we have $0 \leqslant s_0(s) \leqslant f(x)$.*

Finally we show that s_0 is a maximal spline. For if s is a spline satisfying $s_0(x) \leqslant s(x) \leqslant f(x)$ for all x, then $(s_0 - s)(x) = 0$ for x in $[0, h]$ and

$(s_0 - s)((2i + \frac{1}{2})h) = (s_0 - s)'((2i + \frac{1}{2})h) = 0$ for $1 \leqslant i \leqslant M$. It follows then from Eqs. (2.1) and (2.2) that all the B-spline coefficients of $s_0 - s$ must vanish; hence $s_0 = \tilde{s}$ and s_0 is maximal. We summarize

THEOREM 3.5. *Let $f(x) = x^3$ on $[0, 1]$. Let s_0 be the unique quadratic spline with knots at ih for $0 \leqslant i \leqslant N$ such that $s_0(x) = 0$ for s in $[0, h]$ and $s_0((2i + \frac{1}{2})h) - f((2i + \frac{1}{2})h) = s_0'((2i + \frac{1}{2})h) - f'((2i + \frac{1}{2})h) = 0$ for $1 \leqslant i \leqslant M$, where $N = 2M + 1 = 1/h$. Then, with respect to the partial order $\prec$, s_0 is a maximal spline in the set of splines s satisfying $0 \prec s \prec f$, where $s_1 \prec s_2$ if and only if $s_1(x) \leqslant s_2(x)$ on $[0, 1]$, and $\| f - s_0 \|_\infty$ is precisely of order h^2 as h tends to zero.*

This result then shows that a maximal spline need not give an error of the order of best *unconstrained* approximation. We have not been able to discover, however, whether or not best constrained-approximation error is of the same order as for unconstrained approximation.

APPENDIX

Here we shall give some of the tedious and uninformative but useful computations leading to the conclusions in the earlier sections. First we treat the lower bounds $\| s - f \|_\infty \geqslant ch^2$ of Theorems 2.3 and 2.4.

Proof of Theorem 2.3. Because of reasons of symmetry we only need treat the lower bound for the case in which $f(x) = x^3$. We know that Eqs. (2.1) and (2.2) hold; the additional condition that $s(\frac{1}{2}h) - f(\frac{1}{2}h) = s'(\frac{1}{2}h) - f'(\frac{1}{2}h) = 0$ gives $a_{-1} + 6a_0 + a_1 = 8f(\frac{1}{2}h)$ and $a_1 - a_{-1} = 2hf'(\frac{1}{2}h)$, from which we then conclude that $a_1 = a_{-1} + 2hf'(\frac{1}{2}h)$ and $a_0 = \frac{4}{3}f(\frac{1}{2}h) - \frac{1}{3}hf'(\frac{1}{2}h) - \frac{1}{3}a_{-1}$. To evaluate a_{2i} and a_{2i+1} we need to evaluate

$$\sum_{j=1}^{n} g_j' = 3h^2 \sum_{j=1}^{n} \left(2j + \frac{1}{2}\right)^2 = 3h^2 \sum_{j=1}^{n} \left(4j^2 + 2j + \frac{1}{4}\right)$$

$$= 3h^2 \left\{4 \cdot \frac{n(n+1)(2n+1)}{6} + 2 \cdot \frac{n(n+1)}{2} + \frac{n}{4}\right\}.$$

We wish to evaluate

$$s(1) = \frac{1}{2}(a_{2M} + a_{2M+1})$$

$$= \frac{1}{2}\left\{-\frac{1}{3}a_1 + \frac{4}{3}g_M - \frac{2h}{3}\sum_{j=1}^{M-1} g_j' - \frac{h}{3}g_M' + a_1 + 2h\sum_{j=1}^{M} g_j'\right\}$$

$$= \frac{1}{3} a_1 + \frac{2}{3} g_M + \frac{2h}{3} \sum_{j=1}^{M-1} g_j' + \frac{5h}{6} g_M' = \frac{1}{3} \left[a_{-1} + 2h \cdot 3 \left(\frac{1}{2} h \right)^2 \right]$$

$$+ \frac{2}{3} \left(1 - \frac{h}{2} \right)^3 + 2h^3 \left\{ 4 \cdot \frac{(M-1) M (2M-1)}{6} \right.$$

$$\left. + 2 \cdot \frac{(M-1) M}{2} + \frac{M-1}{4} \right\} + \frac{5h}{6} \cdot 3 \left(1 - \frac{h}{2} \right)^2.$$

Recalling that $h(2M+1) = 1$, so that $M = \frac{1}{2}(h^{-1} - 1)$, we obtain finally

$$s(1) = \frac{1}{3} a_{-1} + 1 - \frac{7}{12} h^2 + \frac{7}{24} h^3. \tag{A1}$$

Since $f(1) = 1$, we have

$$s(1) - f(1) = \frac{1}{3} a_{-1} - \frac{7}{12} h^2 + \frac{7}{24} h^3. \tag{A2}$$

On the other hand,

$$\begin{aligned} s(0) - f(0) = s(0) &= \frac{1}{2} (a_{-1} + a_0) \\ &= \frac{1}{2} \left(a_{-1} + \frac{4}{3} f\left(\frac{1}{2} h \right) - \frac{1}{3} h f' \left(\frac{1}{2} h \right) - \frac{1}{3} a_{-1} \right) \\ &= \frac{1}{3} a_{-1} - \frac{h^3}{12}. \end{aligned}$$

Since $\| f - s \|_\infty \geqslant \max\{ | f(0) - s(0) |, | f(1) - s(1) | \}$, we have

$$\| f - s \|_\infty \geqslant \max \left\{ \left| \frac{1}{3} a_{-1} - \frac{h^3}{12} \right|, \left| \frac{1}{3} a_{-1} - \frac{7}{12} h^2 + \frac{7}{24} h^3 \right| \right\}. \tag{A3}$$

This maximum is minimized by letting

$$\frac{1}{3} a_{-1} = \frac{1}{2} \left[\left(\frac{h^3}{12} \right) + \left(\frac{7}{12} h^2 - \frac{7}{24} h^3 \right) \right] \qquad \text{so that} \quad a_{-1} = \frac{7}{8} h^2 - \frac{5}{16} h^3,$$

and we find that

$$\| f - s \|_\infty \geqslant \left| \frac{7}{24} h^2 - \frac{3}{16} h^3 \right| \geqslant \frac{1}{10} h^2 \qquad \text{for} \quad h \leqslant 1. \tag{A4}$$

This completes the proof of Theorem 2.3.

Proof of Theorem 2.4. We still know that Eqs. (2.1) and (2.2) are valid, but we now have $N = 2M$ and we only allow $1 \leqslant i \leqslant M - 1$. Just as in the proof of Theorem 2.3, the interpolation conditions at $\frac{1}{2}h$ give us $a_1 = a_{-1} + 2hf'(\frac{1}{2}h)$ and $a_0 = \frac{4}{3}f(\frac{1}{2}h) - \frac{1}{3}hf'(\frac{1}{2}h) - \frac{1}{3}a_{-1}$. From Eqs. (2.1) and (2.2) we have formulas for $a_2, a_3, \ldots, a_{2M-2}, a_{2M-1}$ in terms of a_1 and hence in terms of a_{-1}; only a_{2M} remains unrepresented. We settle this easily, since the condition $s(1) = f(1) = 1$ gives $\frac{1}{2}(a_{2M-1} + a_{2M}) = 1$ and hence $a_{2M} = 2 - a_{2M-1} = 2 - a_1 - 2h \sum_{j=1}^{M-1} g_j'$. Again from the proof of Theorem 2.3 we know that $s(0) = (1/3)\, a_{-1} - (h^3/12)$, and so the condition $s(0) = f(0) = 0$ finally gives $a_{-1} = h^3/4$. We now proceed to evaluate $s(1 - (h/2)) - f(1 - (h/2))$ and use the fact that $\| s - f \|_\infty \geqslant | s(1 - (h/2)) - f(1 - (h/2))|$. We know that

$$
\begin{aligned}
s\left(1 - \frac{h}{2}\right) &= \frac{1}{8}\,(a_{2M-2} + 6a_{2M-1} + a_{2M}) \\
&= \frac{1}{8}\left\{\left[-\frac{1}{3}a_1 + \frac{4}{3}g_{M-1} - \frac{2h}{3}\sum_{j=1}^{M-2} g_j' - \frac{h}{3}g_{M-1}'\right]\right. \\
&\qquad \left. + 6\left[a_1 + 2h\sum_{j=1}^{M-1} g_j'\right] + \left[2 - a_1 - 2h\sum_{j=1}^{M-1} g_j'\right]\right\} \\
&= \frac{7}{12}a_1 + \frac{1}{6}g_{M-1} + \frac{7}{6}h\sum_{j=1}^{M-1} g_j' + \frac{1}{24}hg_{M-1}' + \frac{1}{4}
\end{aligned}
$$

and we also have $a_1 = a_{-1} + 2hf'((1/2)h) = (h^3/4) + 2h \cdot 3(h^2/4) = (7/4)\, h^3$. Using our formula for $\sum_{j=1}^{M-1} g_j'$ from the proof of Theorem 2.3, we obtain

$$
\begin{aligned}
s\left(1 - \frac{h}{2}\right) &= \frac{7}{12}\left(\frac{7}{4}h^3\right) + \frac{1}{6}\left[\left(2M - \frac{3}{2}\right)h\right]^3 + \frac{7}{6}h \\
&\quad \cdot 3h^2\left\{4 \cdot \frac{(M-1)(M)(2M-1)}{6} + 2\frac{(M-1)\,M}{2} + \frac{M-1}{4}\right\} \\
&\quad + \frac{1}{24}h \cdot 3\left[\left(2M - \frac{3}{2}\right)h\right] + \frac{1}{4}.
\end{aligned}
$$

Recalling that $2Mh = 1$ so that $M = \frac{1}{2}h^{-1}$, we obtain finally

$$
s\left(1 - \frac{h}{2}\right) = 1 - \frac{3}{2}h + \frac{29}{48}h^2 + \frac{41}{16}h^3. \tag{A5}
$$

We of course have

$$
f\left(1 - \frac{h}{2}\right) = \left(1 - \frac{h}{2}\right)^3 = 1 - \frac{3}{2}h + \frac{3}{4}h^2 - \frac{h^3}{8},
$$

so that

$$
s\left(1 - \frac{h}{2}\right) - f\left(1 - \frac{h}{2}\right) = \frac{-7}{48}h^2 + \frac{43}{16}h^3, \tag{A6}
$$

and hence

$$\| s - f \|_\infty \geqslant \left| \frac{7}{48} h^2 - \frac{43}{16} h^3 \right| \geqslant 0.1h^2 \qquad \text{for} \quad h \leqslant 0.01. \tag{A7}$$

This completes the proof of Theorem 2.4.

Proof of Lemma 3.3. Using Eqs. (3.1) and (3.2) we have

$$s_0(1) = \tfrac{1}{2}(a_{2M} + a_{2M+1}) = \tfrac{2}{3}h^3(2M + \tfrac{1}{2})^3 + \tfrac{5}{2}h^3(2M + \tfrac{1}{2})^2 + 2h^3 \sum_{j=1}^{M-1} (2j + \tfrac{1}{2})^2.$$

Recalling that $(2M + 1)h = 1$ and using the expression found in the proof of Theorem 2.3 for the complicated summation, we obtain

$$\begin{aligned} s_0(1) = {} & \frac{2}{3}\left(1 - \frac{h}{2}\right)^3 + \frac{5}{2} h \left(1 - \frac{h}{2}\right)^2 \\ & + 2h^3 \left\{4 \cdot \frac{(M-1)\, M(2M-1)}{6} + 2 \cdot \frac{(M-1)\, M}{2} + \frac{M-1}{4}\right\} \end{aligned}$$

and thence $s_0(1) = 1 - (1/12)\, h^2 - (17/24)\, h^3$. Since $f(1) = 1$, we obtain $f(1) - s_0(1) = (1/12)\, h^2 + (17/24)\, h^3$ as asserted by the lemma, so that certainly $\| f - s_0 \|_\infty$ is *at least* of order h^2. To see that it is *precisely* of this order, we let $\bar{s}$ be the interpolating spline of Theorem 2.3, with B-spline coefficients $\bar{a}_i$; we know that $\| \bar{s} - f \|_\infty = \mathcal{O}(h^2)$, and we will now show that $\| \bar{s} - s_0 \|_\infty = \mathcal{O}(h^3)$ so that $\| s_0 - f \|_\infty = \mathcal{O}(h^2)$, proving our lemma. To do this we note that $\bar{s} - s_0 = \sum_{i=-1}^{N} (\bar{a}_i - a_i)\, B_i$. From the proof of Theorem 2.3 we know that $\bar{s}(0) = (1/3)\, \bar{a}_{-1} - (h^3/12) = 0$, so that $\bar{a}_{-1} = h^3/4$ and hence, again from that proof, $\bar{a}_0 = -h^3/6$ and $\bar{a}_1 = (7/4)\, h^3$. Since $a_{-1} = a_0 = a_1 = 0$, we have $\bar{a}_{-1} - a_{-1} = h^3/4$, $\bar{a}_0 - a_0 = -h^3/6$, and $\bar{a}_1 - a_1 = (7/4)\, h^3$; since Eqs. (2.1) and (2.2) are valid for both the $\bar{a}_i$ and the a_i , by subtraction we immediately find that $\bar{a}_{2i} - a_{2i} = -(1/3)(\bar{a}_1 - a_1) = -(7/12)\, h^3$ and $\bar{a}_{2i+1} - a_{2i+1} = \bar{a}_1 - a_1 = (7/4)\, h^3$ for $1 \leqslant i \leqslant M$. Thus $\| \bar{s} - s_0 \|_\infty \leqslant \|(\bar{a}_{-1} - a_{-1}, \bar{a}_0 - a_0, \ldots, \bar{a}_N - a_N)\|_\infty = (7/4)h^3$ and the proof of Lemma 3.3 is now complete.

Proof of Lemma 3.4. From Eq. (3.2), we have

$$\begin{aligned} a_{2i} & = \frac{4}{3} h^3 \left(2i + \frac{1}{2}\right)^3 - h^3 \left(2i + \frac{1}{2}\right)^2 - 2h^3 \sum_{j=1}^{i-1} \left(2j + \frac{1}{2}\right)^2 \\ & = \frac{4}{3} h^3 \left(2i + \frac{1}{2}\right)^3 - h^3 \left(2i + \frac{1}{2}\right)^2 \\ & \quad - 2h^3 \left\{4 \cdot \frac{(i-1)\, i(2i-1)}{6} + 2 \cdot \frac{(i-1)\, i}{2} + \frac{i-1}{4}\right\} \end{aligned}$$

and hence

$$a_{2i} = h^3(8i^3 + 6i^2 + (1/6)i + (5/12)) \qquad \text{for} \quad 1 \leqslant i \leqslant M. \tag{A8}$$

Similarly using Eq. (3.1) we calculate

$$a_{2i+1} = h^3(8i^3 + 18i^2 + (23/2)i) \qquad \text{for} \quad 1 \leqslant i \leqslant M. \tag{A9}$$

From Eqs. (A8) and (A9) and the fact that $a_{-1} = a_0 = a_1 = 0$, we see that $a_i \geqslant 0$ for all i; since $B_i(x) \geqslant 0$ for all i and x, we immediately see that $s_0(x) \geqslant 0$ for all x as required by the lemma. It remains then only to show that $s_0(x) \leqslant x^3$ for all x. From Eqs. (A8) and (A9) we see that

$$\begin{aligned} s_0(2ih) &= \frac{1}{2}(a_{2i} + a_{2i-1}) = \frac{h^3}{2}\left(8i^3 + 6i^2 + \frac{1}{6}i + \frac{5}{12} + 8i^3 - 6i^2 - \frac{1}{2}i - \frac{3}{2}\right) \\ &= h^3\left(8i^3 - \frac{1}{6}i - \frac{13}{24}\right); \end{aligned}$$

since $f(2ih) = (2ih)^3 = 8i^3h^3$, we have $f(2ih) - s_0(2ih) = h^3((1/6)i + (13/24)) \geqslant 0$, and hence $s_0(2ih) \leqslant f(2ih)$. Similarly we have $s_0((2i+1)h) = (1/2)(a_{2i} + a_{2i+1}) = h^3(8i^3 + 12i^2 + (35/6)i + (5/24))$; since $f((2i+1)h) = [(2i+1)h]^3 = h^3(8i^3 + 12i^2 + 6i + 1)$, we have $f((2i+1)h) - s_0((2i+1)h) = h^3((1/6)i + (19/24)) \geqslant 0$, and hence $s_0((2i+1)h \leqslant f((2i+1)h)$. Thus, we have shown that

$$s_0(ih) \leqslant f(ih) \qquad \text{for all } i. \tag{A10}$$

Clearly we have $s_0(x) \leqslant f(x)$ on $[0, h]$ since $s_0(x) = 0$ there. On $[h, 2h]$, $s_0(x) = a_0B_0(x) + a_1B_1(x) + a_2B_2(x) = a_2B_2(x) = (175/12)\, h^3B_2(x) = (175/12)\, h^3\{(1/2h^2)[x-h]^2\} = (175/24)\, h(x-h)^2$. This gives us $(f - s_0)(h) = h^3 \geqslant 0$, $(f - s_0)(2h) = (17/24)\, h^3 \geqslant 0$, and $(f - s_0)''(x) = 6x - (175/12)h \leqslant 12h - (175/12)h = (-31/12)h < 0$ on $(h, 2h)$. But whenever $g(a) \geqslant 0$, $g(b) \geqslant 0$, and $g''(x) \leqslant 0$ on (a, b), we always can conclude that $g(x) \geqslant 0$ on $[a, b]$. Thus we conclude in our case that $(f - s_0)(x) \geqslant 0$ on $[h, 2h]$ as desired.

We now consider the interval $[(2i+1)h, (2i+2)h]$ for $1 \leqslant i \leqslant M - 1$; in this interval we have $s_0(x) = a_{2i}B_{2i}(x) + a_{2i+1}B_{2i+1}(x) + a_{2i+2}B_{2i+2}(x)$. From Eq. (A10) we already have $(f - s_0)((2i+1)h) \geqslant 0$ and $(f - s_0)((2i+2)h) \geqslant 0$. In $((2i+1)h, (2i+2)h)$ we have $s''(x) = a_{2i}(1/h^2) + a_{2i+1}(-2/h^2) + a_{2i+2}(1/h^2) = h[(8i^3 + 6i^2 + (1/6)i + (5/12)) - 2(8i^3 + 18i^2 + (23/2)i) + (8i^3 + 30i^2 + (217/6)i + (175/2))] = h((40/3)i + 15)$. Therefore $(f - s_0)''(x) = 6x - h((40/3)i + 15) \leqslant 6(2i+2)h - h((40/3)i + 15) = h((-4/3)i - 3) < 0$ on $((2i+1)h, (2i+2)h)$, and thus we conclude as above that $(f - s_0)(x) \geqslant 0$ on $[(2i+1)h, (2i+2)h]$ as desired.

All that remains is to consider the interval $[(2ih, (2i+1)h)]$ for $1 \leqslant i \leqslant M$. We know that $(f - s_0)((2i+\frac{1}{2})h) = (f - s_0)'((2i+\frac{1}{2})h) = 0$; therefore in this interval the Taylor's series gives $(f - s_0)(x) = \frac{1}{2}(f - s_0)''((2i+\frac{1}{2})h) \times [x - (2i+\frac{1}{2})h]^2 + \frac{1}{6}(f - s_0)'''((2i+\frac{1}{2})h)[x - (2i+\frac{1}{2})h]^3$. For $i = 1$, we have $s_0(x) = a_1B_1(x) + a_2B_2(x) + a_3B_3(x) = a_2B_2(x) + a_3B_3(x)$, so that $s_0''(x) = a_2(-2/h^2) + a_3(1/h^2) = (175/12)h^3(-2/h^2) + (79/2)h^3(1/h^2) = (31/3)h$ and $s_0'''(x) = 0$. The Taylor's series yields $(f - s_0)(x) = (1/2)[6 \cdot (5h/2) - (31/3)h][x - (5/2)h]^2 + (1/6)(6)[x - (5/2)h]^3 = (x - (5/2)h)^2[(14/3)h - - (x - (5/2)h)] \geqslant (x - (5/2)h)^2[(14/3)h - (1/2)h] \geqslant 0$ on $(2h, 3h)$, and hence $(f - s_0)(x) \geqslant 0$ on $[2ih, (2i+1)h]$ for $i = 1$. For $i \geqslant 2$, we have $s_0(x) = a_{2i-1}B_{2i-1}(x) + a_{2i}B_{2i}(x) + a_{2i+1}B_{2i+1}(x)$, so that $s_0''(x) = a_{2i-1}(1/h^2) + a_{2i}(-2/h^2) + a_{2i+1}(1/h^2) = h[(8i^3 - 6i^2 - (1/2)i - (3/2)) - 2(8i^3 + 6i^2 + (1/6)i + (5/12)) + (8i^3 + 18i^2 + (23/2)i)] = h((32/3)i - (7/3))$ and $s'''(x) = 0$. The Taylor's series yields $(f - s_0)(x) = (1/2)[6(2i + (1/2))h - h((32/3)i - (7/3))][x - (2i + (1/2))h]^2 + (1/6)(6)[x - (2i + (1/2))h]^3 = [x - (2i + (1/2))h]^2\{h((2/3)i + (8/3)) - [x - (2i + (1/2))h]\} \geqslant [x - (2i + (1/2))h]^2 \times \{h((2/3)i + (8/3)) - (h/2)\} \geqslant 0$ on $(2ih, (2i+1)h)$, and hence $(f - s_0)(x) \geqslant 0$ on $[2ih, (2i+1)h]$. This completes the proof of Lemma 3.4.

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